JOURNAL OF APPROXIMATION THEORY 81, 274–295 (1995)

On Transforming a Tchebycheff System into a Strictly Totally Positive System*

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Communicated by Allan Pinkus

Received November 8, 1993; accepted in revised form March 14, 1994

In this paper, we give a method to transform, if possible, a Tchebycheff basis into a strictly totally positive basis. We introduce the concept of a "bicanonical" system, which is used to provide a test for whether a given Tchebycheff space has a strictly totally positive basis. We also prove that a space of functions defined on an adequate domain has a strictly totally positive basis if it has a canonical complete Tchebycheff basis or if the space is extended Tchebycheff. We also study the problem of enlarging the domain of definition of the functions in a Tchebycheff space to obtain a space with a strictly totally basis.

1. INTRODUCTION

Tchebycheff systems (resp., complete Tchebycheff systems) are closely related to solutions of interpolation problems on the space (resp., on some subspaces) generated by the functions and allow us to construct Lagrange (resp., Newton) formulae for the interpolant by means of the interpolation data. Totally positive systems (resp., strictly totally positive systems), i.e., systems of functions whose collocation matrices have nonnegative (resp., strictly positive) minors, also have important applications in computer aided geometric design due to their variation diminishing properties (cf. [6, 5, 2]).

It is well-known (cf. [10, 12, 16]) that a Tchebycheff system on an open set can be transformed into a complete Tchebycheff system and that this result also holds on sets without supremum and without infimum (see [14, 17]). In this paper we continue this process. Our main goal is to obtain a method to transform, if possible, a given Tchebycheff system into a strictly totally positive system. We also provide a test to determine if a given Tchebycheff space has a strictly totally positive basis.

* Research partially supported by DGICYT under Grant PS90-0121.

0021-9045/95 $6.00

In Section 2 we introduce the basic definitions and notation and give some auxiliary results. In Section 3 we introduce the concept of canonical system, which generalizes some concepts studied previously by several authors ([6, 7, 11]). We show how a Tchebycheff system can always be transformed into a canonical Tchebycheff system. We also prove that a canonical and complete Tchebycheff system is always totally positive. This completes the information given by Karlin in [6] and Schumaker in [11], where some particular canonical complete Tchebycheff systems were studied.

In Section 4 we obtain the announced goals. In fact, if the space has a strictly totally positive basis, we show how to transform any canonical Tchebycheff system into a special canonical system which we call a bicanonical system. Theorem 4.4 shows how to modify bicanonical bases of Tchebycheff spaces in order to obtain a (bicanonical) strictly totally positive system. In [3], the bicanonical totally positive bases are called B-bases. In fact, in that paper it was shown how to obtain a B-basis of a space from a given totally positive basis. Furthermore, if $(b_0, ..., b_n)$ is any B-basis, then any other basis $(v_0, ..., v_n) = (b_0, ..., b_n) K$ is totally positive if and only if the matrix of change of basis K is totally positive. Here we give new examples of bicanonical bases such as generalized monomial bases and generalized Bernstein bases in the space of Müntz polynomials.

In Section 5, we provide some examples of common Tchebycheff spaces which possess strictly totally positive bases. This is the case of extended Tchebycheff spaces and spaces of functions whose domain of definition can be extended to obtain a Tchebycheff space [13, 15]. We also deal with the problem of extending the domain of definition of spaces which have a strictly totally basis to spaces having the same property.

2. Definitions and Auxiliary Results

A sequence of functions $(u_0, ..., u_n)$ defined on a totally ordered set S will be called a system of functions. Many characteristics of the vector space generated by a system of functions (in particular when dealing with interpolation problems and sign or zero properties of fucntions) can be derived from the properties of the corresponding *collocation matrices*

$$M\begin{pmatrix} u_0, u_1, ..., u_n \\ t_0, t_1, ..., t_m \end{pmatrix} := (u_j(t_i))_{i=0, ..., m; j=0, ..., n}, \qquad t_0 < t_1 < \cdots < t_m \quad \text{in } S. \tag{2.1}$$

For this reason, we introduce some terminology about matrices.

DEFINITION 2.1. A matrix A is called *totally positive* (TP) if all minors of A are nonnegative. A is called *strictly totally positive* if all minors of A are strictly positive. If a matrix A has all its minors involving the initial consecutive columns positive, then A is called *lowerly strictly totally positive* (LSTP) (see [4]).

Following the terminology of [12] we introduce the following definitions:

DEFINITION 2.2. A sequence of functions $(u_0, ..., u_n)$ defined on a totally ordered set S is called a *weak Tchebycheff system* (WT system) if

$$\det M\begin{pmatrix} u_0, ..., u_n \\ t_0, ..., t_n \end{pmatrix} \geqslant 0, \qquad t_0 < t_1 < \cdots < t_n \in S. \tag{2.2}$$

If the inequality (2.2) is strict then $(u_0, ..., u_n)$ is called a *Tchebycheff system* (T system). A system $(u_0, ..., u_n)$ is called a *complete Tchebycheff system* (CT system) if $(u_0, ..., u_k)$ is a T system for each $k = 0, 1, ..., n$. Some authors (cf. [17]) use the terminology of *Markov system* instead of CT system. Let us remark that $(u_0, ..., u_n)$ is a CT system if and only if all the collocation matrices (2.1) are LSTP. If $(u_{i_0}, ..., u_{i_k})$ is a T system for all $0 \leqslant i_0 < \cdots < i_k \leqslant n$, $0 \leqslant k \leqslant n$, then $(u_0, ..., u_n)$ is called an *order complete Tchebycheff system*. It can be easily seen that this property is equivalent to saying that all the collocation matrices (2.1) are strictly totally positive and so we shall say that $(u_0, ..., u_n)$ is a *strictly totally positive system* (STP system). In [6] an STP system is also called a Descartes system. If $(u_{i_0}, ..., u_{i_k})$ is a WT system for all $0 \leqslant i_0 < \cdots < i_k \leqslant n$, $0 \leqslant k \leqslant n$, then $(u_0, ..., u_n)$ is called *an order complete weak Tchebycheff system*. It can be shown analogously that all the collocation matrices (2.1) of any order complete weak Tchebycheff system are totally positive and so we shall say that $(u_0, ..., u_n)$ is a *totally positive system* (TP system).

A finite dimensional vector space $\mathscr{U}$ is called a T space (resp., WT space, CT space, TP space, STP space) if $\mathscr{U}$ has a basis which is a T system (resp., WT system, CT system, TP system, STP system). The next result gives a first relationship between these concepts.

PROPOSITION 2.3. *$\mathscr{U}$ is an STP space if and only if it is a T space and a TP space.*

Proof. Clearly every STP space is a T space and a TP space. Conversely, let $(u_0, ..., u_n)$ be any TP basis of a given T space. Let K be any STP matrix of order $n+1$ and let us define $(v_0, ..., v_n) := (u_0, ..., u_n) K$. Taking into account that for any $t_0 < \cdots < t_n$ in S

$$M\begin{pmatrix} v_0, ..., v_n \\ t_0, ..., t_n \end{pmatrix} = M\begin{pmatrix} u_0, ..., u_n \\ t_0, ..., t_n \end{pmatrix} K,$$

we deduce from Theorem 3.1 of [1] that $M\binom{v_0, \ldots, v_n}{t_0, \ldots, t_n}$ is STP because this matrix is the product of a nonsingular TP matrix and an STP matrix. Therefore $(v_0, \ldots, v_n)$ is STP. ▮

We have already mentioned that these spaces have remarkable sign properties as we shall state now.

Given any vector $\lambda = (\lambda_0, \ldots, \lambda_m) \in \mathbb{R}^{m+1}$ let us define

$$\begin{aligned} S^-(\lambda) := \max\{k \mid & \text{ there exist } 0 \leqslant i_0 < \cdots < i_k \leqslant m, \text{ such that} \\ & (-1)^j \lambda_{i_j} > 0 \text{ for all } j \in \{0, \ldots, k\} \text{ or} \\ & (-1)^j \lambda_{i_j} < 0 \text{ for all } j \in \{0, \ldots, k\}\}, \\ S^+(\lambda) := \max\{k \mid & \text{ there exist } 0 \leqslant i_0 < \cdots < i_k \leqslant m, \text{ such that} \\ & (-1)^j \lambda_{i_j} \geqslant 0 \text{ for all } j \in \{0, \ldots, k\} \text{ or} \\ & (-1)^j \lambda_{i_j} \leqslant 0 \text{ for all } j \in \{0, \ldots, k\}\}. \end{aligned}$$

Analogously for any function $u: S \to \mathbb{R}$ we define

$$\begin{aligned} S^-(u) := \sup\{k \mid & \text{ there exist } t_0 < \cdots < t_k \text{ in } S, \\ & \text{such that } S^-(u(t_0), \ldots, u(t_k)) = k\}, \\ S^+(u) := \sup\{k \mid & \text{ there exist } t_0 < \cdots < t_k \text{ in } S, \\ & \text{such that } S^+(u(t_0), \ldots, u(t_k)) = k\}, \\ Z(u) := \#\{t \in S \mid & \; u(t) = 0\}. \end{aligned}$$

T spaces (resp., WT spaces) can be characterized in terms of sign properties as shown in Lemma 3.1 of [17]. Let us restate this result using our terminology.

LEMMA 2.4. *Let $\mathscr{U}$ be an $(n+1)$-dimensional vector space of functions. Then the following properties are equivalent*:

(i) *$\mathscr{U}$ is a T space,*

(ii) *$Z(u) \leqslant n$, $\forall u \neq 0$, and $S^-(u) \leqslant n$, $\forall u \in \mathscr{U}$,*

(iii) *$S^+(u) \leqslant n$, $\forall u \neq 0$.*

Lemma 4.1 of [17] also gives the following characterization of WT spaces:

LEMMA 2.5. *Let $\mathscr{U}$ be an $(n+1)$-dimensional vector space of functions. Then $\mathscr{U}$ is a WT space if and only if $S^-(u) \leqslant n$, $\forall u \in U$.*

It is well-known that any T system can be seen as a system of functions defined on a subset of $\mathbb{R}$ as shown in Theorem 3.3 of [17].

Let $\mathscr{D}$ be the set of all subsets of $\mathbb{R}$ which contain neither their infimum nor their supremum. Analogously to [14], we shall deal only with systems of functions defined on sets $D \in \mathscr{D}$. For these kinds of domains, it was shown in Theorem 3 of [14] that the concepts of T space and CT space coincide. Here we prove that any totally positive T system is an STP system provided that the domain of definition is some $D \in \mathscr{D}$.

PROPOSITION 2.6. *Let $(u_0, ..., u_n)$ be a T system of functions defined on $D \in \mathscr{D}$. Then the following conditions are equivalent:*

(i) *$(u_0, ..., u_n)$ is TP,*

(ii) *$(u_i, ..., u_{i+k})$ is a T system for all $i \in \{0, ..., n\}$ and all $k \in \{0, ..., n-i\}$,*

(iii) *$(u_0, ..., u_n)$ is STP.*

Proof. (i) $\Rightarrow$ (ii) Let us see that if $(u_0, ..., u_n)$ is TP then any collocation matrix

$$B = M\begin{pmatrix} u_i, ..., u_{i+k} \\ s_0, ..., s_k \end{pmatrix}, \qquad s_0 < \cdots < s_k \in D,$$

of $(u_i, ..., u_{i+k})$ has positive determinant. Let us denote by $t_{i+j} := s_j$, $j = 0, ..., k$. Since $D \in \mathscr{D}$ we can take $t_0 < t_1 < \cdots < t_{i-1} < s_0$ and $s_k < t_{i+k+1} < \cdots < t_n$ in D. Then B can be seen as a principal submatrix of the nonsingular TP matrix $M\binom{u_0, ..., u_n}{t_0, ..., t_n}$. By Corollary 3.8 of [1], $\det B > 0$.

(ii) $\Rightarrow$ (iii) Let $A := M\binom{u_0, ..., u_n}{t_0, ..., t_n}$, $t_0 < \cdots < t_n$, be any collocation matrix of $(u_0, ..., u_n)$. By (ii) the minors of A with consecutive columns are strictly positive. By Fekete's lemma (cf. Theorem 2.5 of [1]), A is STP and (iii) follows.

(iii) $\Rightarrow$ (i) Obvious. ∎

3. TRANSFORMING TCHEBYCHEFF SYSTEMS INTO CANONICAL SYSTEMS

The process of transforming a T system into an STP system can be subdivided into several stages. In this section we shall discuss the first stage of this procedure which leads to some special systems. We shall refer to these systems as canonical systems. Canonical systems generalize some systems studied previously by several authors.

DEFINITION 3.1. A *canonical* system $(u_0, ..., u_n)$ of functions defined on a set $D \in \mathscr{D}$ is a system of functions which satisfies

$$\lim_{t \to a} \frac{u_i(t)}{u_{i-1}(t)} = 0, \qquad i = 1, ..., n, \tag{3.1}$$

where $a = \inf D$.

Remark 3.2. Let us observe that from Lemma 2.4 any function u in a T space $\mathscr{U}$ has a bounded number of sign changes. Thus u has constant sign on a neighbourhood of a. On the other hand, if $u, w \in \mathscr{U}$ and $w \neq 0$ then $\lim_{t \to a} u(t)/w(t) \in \mathbb{R} \cup \{-\infty, +\infty\}$. In fact, given any $\alpha \in \mathbb{R}$, $u/w - \alpha = (u - \alpha w)/w$ has constant sign on a neighbourhood of a. Therefore one of the three following possibilities occurs on a neighbourhood of a: $u(t)/w(t) > \alpha$, $u(t)/w(t) = \alpha$, or $u(t)/w(t) < \alpha$. Then, if

$$A = \{\alpha \in \mathbb{R} \mid \text{there exists a neighbourhood } V \text{ of } a \text{ such that } u/w \leqslant \alpha \text{ on } V\},$$

it can be easily shown that $\lim_{t \to a} u(t)/w(t) = \inf A$, with the convention $\inf \emptyset = +\infty$.

The following result will be very useful.

LEMMA 3.3. *Let $(u_0, ..., u_n)$ be a T system of functions defined on $D \in \mathscr{D}$ and $a := \inf D$. Let us assume that $u_0 > 0$ on a neighbourhood of a and*

$$\lim_{t \to a} \frac{u_i(t)}{u_0(t)} = 0, \qquad \text{for} \quad i = 1, ..., n.$$

Then $(u_1, ..., u_n)$ is a T system.

Proof. We first prove that $(u_1, ..., u_n)$ is a WT system. Let $t_1 < \cdots < t_n$ in D. For any $t < t_1$

$$\frac{1}{u_0(t)} \det M\begin{pmatrix} u_0, u_1, ..., u_n \\ t, t_1, ..., t_n \end{pmatrix} = \det \begin{pmatrix} 1 & \dfrac{u_1(t)}{u_0(t)} & \cdots & \dfrac{u_n(t)}{u_0(t)} \\ u_0(t_1) & u_1(t_1) & \cdots & u_n(t_1) \\ \vdots & & & \vdots \\ u_0(t_n) & u_1(t_n) & \cdots & u_n(t_n) \end{pmatrix}.$$

Taking limits as $t \to a$, we obtain

$$0 \leqslant \det \begin{pmatrix} 1 & 0 \cdots 0 \\ u_0(t_1) & \\ \vdots & M\begin{pmatrix} u_1, \ldots, u_n \\ t_1, \ldots, t_n \end{pmatrix} \\ u_0(t_n) & \end{pmatrix} = \det M\begin{pmatrix} u_1, \ldots, u_n \\ t_1, \ldots, t_n \end{pmatrix}.$$

So, $(u_1, \ldots, u_n)$ is a WT system. By Lemma 2.4 it remains to prove that $Z(u) \leqslant n-1$, for all $u \neq 0$, in $\operatorname{span}(u_1, \ldots, u_n)$. Let us assume that $Z(u) \geqslant n$ for some $u \in \operatorname{span}(u_1, \ldots, u_n)$, $u \neq 0$, and we shall obtain a contradiction. Since $u \in \operatorname{span}(u_0, u_1, \ldots, u_n)$ and $(u_0, u_1, \ldots, u_n)$ is a T system, by Lemma 2.4, we have $Z(u) = n$. Let $\tau_1 < \cdots < \tau_n$ be the zeros of u. Let $\tau_{n+1} \in D$ be such that $\tau_{n+1} > \tau_n$. Without loss of generality we may assume that $u > 0$ on a neighbourhood of a. Let $v \in \operatorname{span}(u_0, u_1, \ldots, u_n)$ be the solution of the interpolation problem $v(\tau_i) = (-1)^i$, $i = 1, \ldots, n+1$. So $S^-(v) \geqslant n$ and since $(u_1, \ldots, u_n)$ is a WT system, Lemma 2.5 implies that $v \notin \operatorname{span}(u_1, \ldots, u_n)$. On the other hand $(u_0, \ldots, u_n)$ is a T system and by Lemma 2.4 (iii), $S^+(v) \leqslant n$, which implies that $v(t) < 0$ for all $t \in D$, $t < \tau_1$. Therefore $\alpha := \lim_{t \to a} v(t)/u_0(t) \leqslant 0$ and $\alpha \neq 0$ because $v \notin \operatorname{span}(u_1, \ldots, u_n)$.

By our assumption, $u > 0$ on a neighbourhood of a. Let $\tau_0 \in D$, $\tau_0 < \tau_1$ such that $u(\tau_0) > 0$. Let $0 < \varepsilon < -u(\tau_0)/v(\tau_0)$ and $w = u + \varepsilon v$. Since $\lim_{t \to a} w(t)/u_0(t) = \varepsilon\alpha < 0$, there exists $\tau_{-1} \in D$, $\tau_{-1} < \tau_0$ such that $w(\tau_{-1}) < 0$. By the choice of ε, $w(\tau_0) > 0$. Thus

$$S^-(w) \geqslant S^-(w(\tau_{-1}), w(\tau_0), w(\tau_1), \ldots, w(\tau_n)) = n+1,$$

which contradicts Lemma 2.4 (ii). ∎

Let us remark that, omitting the assumption $u_0 > 0$ on a neighbourhood of a in the previous lemma, we can obtain the statement $(u_1, \ldots, u_n)$ or $(-u_1, \ldots, u_n)$ is a T system.

Let us observe that canonical systems satisfy some properties "at the left of D." It will be useful to consider analogous properties to the right, that is,

$$\lim_{t \to b} \frac{u_{i-1}(t)}{u_i(t)} = 0, \qquad i = 1, \ldots, n, \tag{3.2}$$

where $b := \sup D$. This can be equivalently stated as saying that the system $(u_0, \ldots, u_n)^{\#}$ given by

$$(u_0, \ldots, u_n)^{\#}(s) := (u_n, \ldots, u_0)(-s), \qquad s \in -D, \tag{3.3}$$

is canonical.

From the previous definition, it follows that $(u_0, ..., u_n)$ is a T system if and only if $(u_0, ..., u_n)^{\#}$ is a T system. Since a CT system $(u_0, ..., u_n)$ is a system of functions whose collocation matrices have all its minors with initial columns strictly positive, we obtain that $(u_0, ..., u_n)^{\#}$ is CT if and only if any collocation matrix of $(u_0, ..., u_n)$ has all its minors with final columns strictly positive. This will be used in the proof of the following proposition.

PROPOSITION 3.4. *If $(u_0, ..., u_n)$ is a canonical T system and $u_i > 0$ on a neighbourhood of $a = \inf D$, $i = 0, ..., n-1$, then $(u_0, ..., u_n)^{\#}$ is a* CT *system.*

Proof. We prove this result by induction on n. If $n = 0$, it is trivial. Let us assume that the result holds for $n-1$, and let us prove it for n. The system $(u_1, ..., u_n)$ is clearly canonical and, by Lemma 3.3, it is also a T system. By the induction hypothesis $(u_1, ..., u_n)^{\#}$ is CT and since $(u_0, ..., u_n)^{\#}$ is a T system, $(u_0, ..., u_n)^{\#}$ is a CT system. ∎

The following result gives a sufficient condition for a system to be STP.

PROPOSITION 3.5. *If $(u_0, ..., u_n)$ is a canonical CT system, then $(u_0, ..., u_n)$ is STP.*

Proof. Let us first prove by induction on i that $(u_i, u_{i+1}, ..., u_k)$ is a T system for all $k \geqslant i$. Since $(u_0, ..., u_n)$ is a CT system, $(u_0, ..., u_k)$ is a T system for all $k \geqslant 0$ and the result follows for $i = 0$. We now assume that $(u_{i-1}, ..., u_k)$ is a T system for all $k \geqslant i-1$. In particular, u_{i-1} is a T system and therefore a positive function. For a given $k \geqslant i-1$, since $(u_{i-1}, ..., u_k)$ is Tchebycheff, canonical, and $u_{i-1} > 0$, by Lemma 3.3 $(u_i, ..., u_k)$ is a T system.

Thus we have shown that $(u_i, ..., u_k)$ is a T system for any $i = 0, ..., n$, $k = i, ..., n$. The proposition follows from Proposition 2.6. ∎

Our definition of canonical CT system generalizes the concept which other authors have called "canonical complete Tchebycheff system" (CCT). In [11], a canonical CT system is a system of functions $(u_0, ..., u_n)$ defined on an interval $[a, b]$ such that

$$u_1(t) = u_0(t) \int_a^t d\sigma_1(s_1)\, ds_1,$$

$$\ldots$$

$$u_n(t) = u_0(t) \int_a^t \cdots \int_a^{s_{n-1}} d\sigma_n(s_n) \cdots d\sigma_1(s_1),$$

where u_0 is a bounded positive function and $\sigma_1, ..., \sigma_n$ are bounded, right continuous, monotone increasing functions on $[a, b]$.

It is easy to check that, under these conditions, $(u_0, \ldots, u_n)$ is canonical and CT on (a, b) in the sense defined in this paper. By the previous proposition such a system is STP on (a, b). Furthermore, taking into account Lemma 2.1 of [11], we may deduce that any CCT system $(u_0, \ldots, u_n)$ in the sense of Schumaker is STP on $(a, b]$ and, since $u_0(a) > 0$, $u_1(a) = \cdots = u_n(a) = 0$, we may deduce that $(u_0, \ldots, u_n)$ is TP on $[a, b]$.

We finish this section showing that any T system can be transformed into a canonical T system.

THEOREM 3.6. *Any Tchebycheff space* $\mathcal{U}$ *of functions defined on* $D \in \mathcal{D}$ *has a canonical Tchebycheff basis* $(v_0, \ldots, v_n)$ *with* $v_i > 0$ *on a neighbourhood of* $a = \inf D$.

Proof. Let $(u_0, \ldots, u_n)$ be a basis of $\mathcal{U}$ which is a T system. From Remark 3.2 it is clear that we may choose $i \in \{0, \ldots, n\}$ such that $\lim_{t \to a} |u_j/u_i| \neq \infty$, $j = 0, \ldots, n$. If ε is the sign of u_i on a neighbourhood of a, then $(\varepsilon u_i, u_1, \ldots, -\varepsilon u_0, u_{i+1}, \ldots, u_n)$ is a T system. Let $w_0 = \varepsilon u_i$, $w_j = u_j - \lim_{t \to a} (u_j(t)/u_i(t))\, u_i$, $j \in \{1, \ldots, n\} \backslash \{i\}$ and $w_i = -\varepsilon(u_0 - \lim_{t \to a} (u_0(t)/u_i(t))\, u_i)$. Then $(w_0, \ldots, w_n)$ is a T system such that $w_0 > 0$ on a neighbourhood of a and $\lim_{t \to a} w_i(t)/w_0(t) = 0$, $i = 1, \ldots, n$. By Lemma 3.3 $(w_1, \ldots, w_n)$ is also a T system and, applying iteratively to this system the previous construction, the result follows. ∎

Remark 3.7. The previous theorem together with Proposition 3.4 provides an alternative proof of the following well-known fact: A T space of functions defined on an open set has a CT basis (cf. [14]).

4. TRANSFORMING CANONICAL TCHEBYCHEFF SYSTEMS INTO BICANONICAL SYSTEMS

In the previous section we transformed a T system into a canonical T system. This is the first step in transforming "if possible" a T system into an STP system. We now transform a canonical system into another canonical system which enjoys further properties.

PROPOSITION 4.1. *Let* $(u_0, \ldots, u_n)$ *be a canonical Tchebycheff basis of a space* $\mathcal{U}$ *of functions defined on* $D \in \mathcal{D}$ *and* $b = \sup D$. *Then there exists a lower triangular matrix* L *with unit diagonal and a permutation* σ *of* $\{0, 1, \ldots, n\}$ *such that* $(v_0, \ldots, v_n) = (u_0, \ldots, u_n) L$ *is a canonical Tchebycheff basis of* $\mathcal{U}$ *satisfying*

$$\lim_{t \to b} \frac{v_{\sigma(i)}(t)}{v_{\sigma(i+1)}(t)} = 0, \qquad i = 0, \ldots, n-1.$$

That is, $(v_{\sigma(0)}, \ldots, v_{\sigma(n)})^{\#}$ *is canonical.*

Proof. From Remark 3.2 the set

$$\left\{ i \in \{0, 1, ..., n\} \;\middle|\; \lim_{t \to b} \left| \frac{u_j(t)}{u_i(t)} \right| \neq \infty, \quad \forall j = 0, ..., n \right\}$$

is nonempty. Let k_0 be the maximum of that set. Let

$$\begin{cases} w_j := u_j, & \forall j \geqslant k_0, \\ w_j := u_j - \lim\limits_{t \to b} \dfrac{u_j(t)}{u_{k_0}(t)} \cdot u_{k_0}, & \text{otherwise.} \end{cases}$$

It is simple to check that $(w_0, ..., w_n)$ is also a canonical Tchebycheff basis of $\mathcal{U}$ and

$$\lim_{t \to b} \frac{w_j(t)}{w_{k_0}(t)} = 0, \qquad \forall j \neq k_0.$$

Let k_1 be the maximum of

$$\left\{ i \in \{0, 1, ..., n\} \backslash \{k_0\} \;\middle|\; \lim_{t \to b} \left| \frac{w_j(t)}{w_i(t)} \right| \neq \infty, \forall j \in \{0, ..., n\} \backslash \{k_0\} \right\}$$

and let us define

$$\begin{cases} z_j := w_j, & \text{if } j = k_0 \text{ or } j \geqslant k_1, \\ z_j := w_j - \lim\limits_{t \to b} \dfrac{w_j(t)}{w_{k_1}(t)} \cdot w_{k_1}, & \text{otherwise.} \end{cases}$$

Again $(z_0, ..., z_n)$ is a canonical Tchebycheff basis of $\mathcal{U}$ and

$$\lim_{t \to b} \frac{z_j(t)}{z_{k_0}(t)} = 0, \forall j \neq k_0 \qquad \text{and} \qquad \lim_{t \to b} \frac{z_j(t)}{z_{k_1}(t)} = 0, \forall j \neq k_0, k_1.$$

Continuing iteratively this procedure we obtain indices $k_0, k_1, ..., k_n$ and a canonical Tchebycheff basis $(v_0, ..., v_n)$ such that

$$\lim_{t \to b} \frac{v_{k_{i+1}}(t)}{v_{k_i}(t)} = 0.$$

Since the functions v_i are obtained by subtracting from u_i linear combinations of the functions u_j for $j > i$, it follows that the matrix of change of basis L is lower triangular with unit diagonal. Defining σ as the permutation given by $\sigma(i) = k_{n-i}$, $i = 0, ..., n$, the result follows. ∎

Remark 4.2. If the canonical system $(v_0, ..., v_n)$ obtained in the previous proposition has associated the permutation $\sigma = 1$ (identity), then $(v_0, ..., v_n)^{\#}$ is also canonical.

This motivates the following defintion:

DEFINITION 4.3. A system of functions $(u_0, ..., u_n)$ is said to be *bicanonical* if both $(u_0, ..., u_n)$ and $(u_0, ..., u_n)^{\#}$ are canonical.

It is well-known that the space $\prod_n$ of polynomials of degree less than or equal to n is a Tchebycheff space on any interval I of $\mathbb{R}$. In [2], it is shown that, for $I = (0, \infty)$, the monomial basis $(1, t, ..., t^n)$ is bicanonical and that, for $I = [0, 1]$, the Bernstein basis is bicanonical. An interesting generalization of the polynomials on $(0, \infty)$ is given by the Müntz polynomials in $(0, \infty)$, which is the space generated by the basis $(t^{\lambda_i})_{i \in \mathbb{N} \cup \{0\}}$, where $(\lambda_i)_{i \in \mathbb{N} \cup \{0\}}$ is a strictly increasing sequence of real numbers. Let us consider the space of Müntz polynomials $\mathcal{M}$ generated by the basis $(t^{\lambda_0}, t^{\lambda_1}, ..., t^{\lambda_n})$. Since the kernel $K(\lambda, t) := t^{\lambda} = \exp(\lambda \log(t))$ is STP in $\mathbb{R} \times (0, \infty)$ (cf. [6, p. 100]), $\mathcal{M}$ is an STP space and, in particular, a T space. In addition, it is easy to check that this basis is bicanonical. The next result shows that the concept of a bicanonical basis of a T space leads to another sufficient condition for a system to be STP.

THEOREM 4.4. *Let $(u_0, ..., u_n)$ be a bicanonical basis of a Tchebycheff space of functions defined on $D \in \mathcal{D}$. Then*:

(i) *u_i has constant strict sign for each $i \in \{0, ..., n\}$.*

(ii) *If $u_0, ..., u_n > 0$ then $(u_0, ..., u_n)$ is STP.*

Proof. Let $a = \inf D$ and $b = \sup D$. First, let us prove (i), assuming that $(u_0, ..., u_n)$ is a T system. Given $i \in \{0, ..., n\}$, let us see that either $u_i > 0$ or $-u_i > 0$. Let $\varepsilon_j \in \{1, -1\}$ be the sign of u_j on a neighbourhood of a. By Lemma 3.3 we deduce that $(\varepsilon_0 u_1, u_2, ..., u_n)$ is a T system. Iterating this procedure we obtain that $(\varepsilon_0 \cdots \varepsilon_{k-1} u_k, u_{k+1}, ..., u_n)$ is a T system, $k = 1, 2, ..., i$. Now, reasoning analogously on the T system $(\varepsilon_0 \cdots \varepsilon_{i-1} u_i, u_{i+1}, ..., u_n)^{\#}$ and taking $\varepsilon'_j \in \{1, -1\}$ as the sign of u_j on a neighbourhood of b, we deduce that $(\varepsilon_0 \cdots \varepsilon_{i-1} u_i, u_{i+1}, ..., u_{k-1}, \varepsilon'_{k+1} \cdots \varepsilon'_n u_k)^{\#}$, $k = n, n-1, ..., i$, is a T system. Finally $\varepsilon_0 \cdots \varepsilon_{i-1} \varepsilon'_{i+1} \cdots \varepsilon'_n u_i > 0$ and thus u_i has constant strict sign on D. The general case follows from the fact that if $(u_0, ..., u_n)$ is a basis of a T space, then either $(u_0, u_1, ..., u_n)$ or $(-u_0, u_1, ..., u_n)$ is a T system.

In order to prove (ii), let $\varepsilon \in \{-1, 1\}$ be such that $(\varepsilon u_0, u_1, ..., u_n)$ is a T system. From Proposition 3.4 and Proposition 3.5, we deduce that $(\varepsilon u_0, u_1, ..., u_n)^{\#}$ is STP. Thus $\varepsilon = 1$ and $(u_0, ..., u_n)$ is STP. ∎

Let us apply the previous theorem to show that the generalization of the Bernstein basis $P=(p_0, ..., p_n)$ given by

$$p_i(t) := (-1)^{n-i} \lambda_{i+1} \cdots \lambda_n t^\lambda[\lambda_i, ..., \lambda_n], \qquad i=0, ..., n-1,$$
$$p_n(t) := t^{\lambda_n},$$

$0=\lambda_0 < \cdots < \lambda_n$, is strictly totally positive on $(0, 1)$. Here $t^\lambda[\lambda_i, ..., \lambda_n]$ denotes the divided difference of the function $\lambda \to t^\lambda$ at the points $\lambda_i, ..., \lambda_n$. If we take $\lambda_i = i$, $i=0, ..., n$, we obtain $p_i(t) = \binom{n}{i}(1-t)^{n-i} t^i$. The basis P of Müntz polynomials was described by Hirschman and Widder to construct a generalized Bernstein operator

$$B_n(f; t) := \sum_{i=0}^{n} f(\alpha_i) p_i(t), \qquad \alpha_i := \left[\left(1-\frac{\lambda_1}{\lambda_{i+1}}\right)\cdots\left(1-\frac{\lambda_1}{\lambda_n}\right)\right]^{1/n},$$
$$i=1, ..., n,$$

whose approximation properties have been further investigated by Leviatan (see [8, 9]). Clearly $M=(t^{\lambda_0}, ..., t^{\lambda_n})$, $t \in (0, 1)$, is an STP basis of the space $\mathscr{P}$ generated by the functions $(p_0(t), ..., p_n(t))$, $t \in (0, 1)$. So $\mathscr{P}$ is an STP space and, in particular, a T space.

Let us show that P is a bicanonical basis of positive functions. First we observe that p_i is a linear combination of $t^{\lambda_i}, ..., t^{\lambda_n}$ with nonzero coefficient in t^{λ_i}, for all i. This implies that $\lim_{t \to 0} p_{i+1}(t)/p_i(t) = 0$, $i=0, ..., n-1$, and so P is canonical. On the other hand, the functions p_i are differentiable at $t=1$ and

$$p_i^{(k)}(1) = (-1)^{n-i} \lambda_{i+1} \cdots \lambda_n \cdot (\lambda(\lambda-1)\cdots(\lambda-k+1))[\lambda_i, ..., \lambda_n],$$

which implies that $p_i^{(k)}(1)=0$ if $k<n-i$ and $(-1)^{n-i} p_i^{(n-i)}(1) > 0$. So p_i has at $t=1$ a zero of multiplicity $n-i$ and thus $\lim_{t \to 1} p_i(t)/p_{i+1}(t) = 0$, $i=0, ..., n-1$. That is, P is bicanonical. Furthermore, from the sign of the derivatives we deduce that p_i is positive on an interval $(1-\delta, 1)$. Applying Theorem 4.4 we deduce that the functions p_i are positive on $(0, 1)$ and that P is an STP basis. The above arguments can be applied even if $\lambda_0 \neq 0$. For this case we choose $p_i(t) := (-1)^{n-i}(\lambda_{i+1}-\lambda_0)\cdots(\lambda_n-\lambda_0)\, t^\lambda[\lambda_i, ..., \lambda_n]$, which form a basis, normalized so that $\sum_{i=0}^{n} p_i(t) = t^{\lambda_0}$, and obtain analogously that it is a bicanonical STP basis of the space.

The following lemma allows us to transform an STP system into another STP system.

LEMMA 4.5. *Let* $(u_0, ..., u_n)$ *be an STP system of functions defined on* $D \in \mathscr{D}$ *and* $a = \inf D$. *Then the system* $(v_0, ..., v_n)$ *given by*

$$\begin{cases} v_0 = u_0, \\ v_i = u_i - \lim\limits_{t \to a} \dfrac{u_i(t)}{u_0(t)} u_0, \qquad i = 1, ..., n, \end{cases}$$

is also an STP system.

Proof. Let us show first that $\lim_{t \to a} u_i(t)/u_0(t)$ is finite. From

$$\begin{vmatrix} u_0(t) & u_i(t) \\ u_0(s) & u_i(s) \end{vmatrix} > 0, \quad s > t, \quad s, t \in D,$$

we derive $u_i(s)/u_0(s) > u_i(t)/u_0(t)$, which implies that u_i/u_0 is strictly increasing. Since $u_i/u_0 > 0$ on D, we deduce that $\lim_{t \to a} u_i(t)/u_0(t)$ exists and it is a nonnegative number.

It is clear that $(v_0, ..., v_n)$ is also Tchebycheff on D. Let us see now that it is TP on D. It remains to see that any collocation matrix $M\binom{v_0, ..., v_n}{t_1, ..., t_n}$, $t_1 < \cdots < t_n$, is TP. For any $t < t_1$ the matrix

$$\begin{pmatrix} 1 \quad \dfrac{u_1(t)}{u_0(t)} \cdots \dfrac{u_n(t)}{u_0(t)} \\ M\begin{pmatrix} u_0, ..., u_n \\ t_1, ..., t_n \end{pmatrix} \end{pmatrix}$$

is TP because it is the matrix obtained by dividing the first row of the TP matrix $M\binom{u_0, ..., u_n}{t, t_1, ..., t_n}$ by the positive number $u_0(t)$. Taking limits when $t \to a$ we may deduce that the matrix

$$A = \begin{pmatrix} 1 \quad \lim\limits_{t \to a} \dfrac{u_1(t)}{u_0(t)} \cdots \lim\limits_{t \to a} \dfrac{u_n(t)}{u_0(t)} \\ M\begin{pmatrix} u_0, ..., u_n \\ t_1, ..., t_n \end{pmatrix} \end{pmatrix}$$

is TP. Performing a step of Gaussian elimination by columns to produce zeros in the first row of A, it follows from Corollary 3.4 of [1] that the resulting matrix

$$\begin{pmatrix} 1 \quad 0 \cdots 0 \\ M\begin{pmatrix} v_0, ..., v_n \\ t_1, ..., t_n \end{pmatrix} \end{pmatrix}$$

is TP and thus $M\binom{v_0, ..., v_n}{t_1, ..., t_n}$ is also TP. Therefore $(v_0, ..., v_n)$ is TP on D and by Proposition 2.6 it is also STP on D. ∎

The following proposition shows that in an STP space any canonical basis allows us to construct STP subspaces of any dimension.

PROPOSITION 4.6. *Let $\mathscr{U}$ be an STP space of functions defined on $D \in \mathscr{D}$. For any canonical basis $(b_0, ..., b_n)$ of $\mathscr{U}$, the subspaces spanned by b_k, $b_{k+1}, ..., b_n$, $k = 0, ..., n$, are STP spaces.*

Proof. Let $(u_0, ..., u_n)$ be any STP basis of $\mathscr{U}$. From Lemma 4.5 we may construct an STP system $(v_0, ..., v_n)$ such that $v_0 = u_0$ and $\lim_{t \to a} v_i(t)/u_0(t) = 0$, $i = 1, ..., n$. Therefore the n-dimensional subspace $\mathscr{U}_1$ generated by $(v_1, ..., v_n)$ is an STP space. Furthermore one has

$$\mathscr{U}_1 = \left\{ u \in \mathscr{U} \;\middle|\; \lim_{t \to a} \frac{u(t)}{u_0(t)} = 0 \right\}.$$

Let us prove that $b_1, ..., b_n \in \mathscr{U}_1$ and that they form a basis of the STP space $\mathscr{U}_1$.

Since $(b_0, ..., b_n)$ is canonical, $\lim_{t \to a} b_i(t)/b_0(t) = 0$, $i = 1, ..., n$. Now if we express b_0 as a linear combination of the basis $(v_0, ..., v_n)$ we may write

$$\lim_{t \to a} \frac{b_0(t)}{u_0(t)} = \lim_{t \to a} \frac{\sum_{i=0}^{n} \lambda_i v_i(t)}{u_0(t)} = \lambda_0$$

and deduce that this limit is finite. Therefore

$$\lim_{t \to a} \frac{b_i(t)}{u_0(t)} = \lambda_0 \lim_{t \to a} \frac{b_i(t)}{b_0(t)} = 0, \qquad i = 1, ..., n.$$

It therefore follows that $\lambda_0 \neq 0$ and $\mathscr{U}_1 = \text{span}(b_1, ..., b_n)$. Thus $(b_1, ..., b_n)$ is again a canonical basis of the STP space $\mathscr{U}_1$. Applying iteratively the previous arguments, the result follows. ∎

The next theorem shows that, in an STP space, the construction of Proposition 4.1 is possible only when the permutation $\sigma = 1$.

THEOREM 4.7. *Let $\mathscr{U}$ be an STP space of functions defined on $D \in \mathscr{D}$. If $(b_0, ..., b_n)$ is a canonical basis of $\mathscr{U}$ such that*

$$\lim_{t \to b} \frac{b_{\sigma(i)}(t)}{b_{\sigma(i+1)}(t)} = 0, \qquad i = 0, ..., n-1, \tag{4.1}$$

for some permutation σ of $\{0, 1, ..., n\}$, then $\sigma = 1$ (identity). Therefore $(b_0, ..., b_n)$ is bicanonical.

Proof. Let us assume that $\sigma(n) \neq n$ and consider the space $\mathscr{W}$ generated by $b_{\sigma(n)}, b_{\sigma(n)+1}, ..., b_n$, which is an STP space by Proposition 4.6. Let us observe that $\lim_{t \to a} b_i(t)/b_{\sigma(n)}(t) = 0$ for each $i \in \{\sigma(n)+1, \sigma(n)+2, ..., n\}$ since $(b_0, ..., b_n)$ is canonical. By (4.1) we also have that $\lim_{t \to b} b_i(t)/b_{\sigma(n)}(t) = 0$ for all $i \neq \sigma(n)$. Any function $w \in \mathscr{W}'$ can be written in the form $w = \sum_{i=\sigma(n)}^{n} \alpha_i b_i$ and satisfies

$$\lim_{t \to a} \frac{w(t)}{b_{\sigma(n)}(t)} = \alpha_{\sigma(n)} = \lim_{t \to b} \frac{w(t)}{b_{\sigma(n)}(t)}. \tag{4.2}$$

Let $(w_{\sigma(n)}, ..., w_n)$ be an STP basis of $\mathscr{W}'$. By Lemma 4.5 we may assume without loss of generality that

$$\lim_{t \to a} \frac{w_i(t)}{w_{\sigma(n)}(t)} = 0, \qquad i = \sigma(n)+1, ..., n. \tag{4.3}$$

From (4.3) and (4.2) we derive

$$\lim_{t \to a} \frac{w_i(t)}{b_{\sigma(n)}(t)} = \lim_{t \to a} \frac{w_i(t)}{w_{\sigma(n)}(t)} \cdot \lim_{t \to a} \frac{w_{\sigma(n)}(t)}{b_{\sigma(n)}(t)} = 0, \qquad i = \sigma(n)+1, ..., n,$$

and taking into account that $(w_{\sigma(n)}, ..., w_n)$ is a basis of $\mathscr{W}'$ we deduce that

$$\lim_{t \to a} \frac{w_{\sigma(n)}(t)}{b_{\sigma(n)}(t)} \neq 0. \tag{4.4}$$

Now, using (4.4), (4.3), and (4.2), we deduce that

$$0 = \lim_{t \to a} \frac{w_i(t)}{w_{\sigma(n)}(t)} = \frac{\lim_{t \to a} \frac{w_i(t)}{b_{\sigma(n)}(t)}}{\lim_{t \to a} \frac{w_{\sigma(n)}(t)}{b_{\sigma(n)}(t)}} = \frac{\lim_{t \to b} \frac{w_i(t)}{b_{\sigma(n)}(t)}}{\lim_{t \to b} \frac{w_{\sigma(n)}(t)}{b_{\sigma(n)}(t)}} = \lim_{t \to b} \frac{w_i(t)}{w_{\sigma(n)}(t)}.$$

If $t \in D$, for any $s > t$, $s \in D$, let us consider the 2×2 collocation matrix $M\binom{w_{\sigma(n)}, w_{\sigma(n)+1}}{t, s}$. Then taking limits we obtain

$$\lim_{s \to b} \frac{1}{w_{\sigma(n)}(t)\, w_{\sigma(n)}(s)} \det M \begin{pmatrix} w_{\sigma(n)}, w_{\sigma(n)+1} \\ t, s \end{pmatrix}$$

$$= \lim_{s \to b} \begin{vmatrix} 1 & \dfrac{w_{\sigma(n)+1}(t)}{w_{\sigma(n)}(t)} \\ 1 & \dfrac{w_{\sigma(n)+1}(s)}{w_{\sigma(n)}(s)} \end{vmatrix} = \frac{-w_{\sigma(n)+1}(t)}{w_{\sigma(n)}(t)} < 0.$$

This means that there must exist collocation matrices with negative determinants. This contradicts the fact that $(w_{\sigma(n)}, ..., w_n)$ is an STP system.

Thus we have shown that $\sigma(n) = n$. Now, applying Proposition 4.6 to the canonical basis $(b_{\sigma(0)}, ..., b_{\sigma(n)})$, we see that the space $\text{span}(b_{\sigma(0)}, ..., b_{\sigma(n-1)}) = \text{span}(b_0, ..., b_{n-1})$ has an STP basis. Iterating the previous procedure we obtain that $\sigma(n-1) = n-1, ..., \sigma(0) = 0$. ∎

As an application of Theorem 4.7, let us prove that the space $\prod_n$ of polynomials of degree less than or equal to n, $n \geqslant 1$, on $\mathbb{R}$ is not an STP space. In fact, the basis $(t^n, ..., t, 1)$ is a canonical basis for which (4.1) holds for the permutation $\sigma(i) = n - i$, $i = 0, ..., n$. From Theorem 4.7 we derive that $\prod_n$ cannot be an STP space on $\mathbb{R}$.

COROLLARY 4.8. *Let $\mathscr{U}$ be a T space of functions defined on $D \in \mathscr{D}$. Then the following properties are equivalent*:

(i) *$\mathscr{U}$ has an STP basis.*

(ii) *$\mathscr{U}$ has a bicanonical basis.*

(iii) *$\mathscr{U}$ has a bicanonical STP basis.*

(iv) *$\mathscr{U}$ has a canonical CT basis.*

Proof. (i) $\Rightarrow$ (ii). This is a consequence of Theorem 3.6, Proposition 4.1, and Theorem 4.7.

(ii) $\Rightarrow$ (iii). Let $(b_0, ..., b_n)$ be a bicanonical basis. By Theorem 4.4(i) b_i has constant sign on D for each i. Let $\varepsilon_i = \text{sign}(b_i)$. Then $(\varepsilon_0 b_0, ..., \varepsilon_n b_n)$ is STP by Theorem 4.4(ii).

(iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (i). By Proposition 3.5 each canonical CT system is STP. ∎

Remark 4.9. Now we may give a construction of STP bases from T systems. In fact, in the proof of Theorem 3.6 it was indicated how to transform any given Tchebycheff basis into a canonical Tchebycheff basis. From Theorem 4.7 we may deduce that, if there exists an STP basis in the space, it can be obtained as suggested in the proof of Proposition 4.1 taking $k_0 = n$, $k_1 = n-1, ..., k_n = 0$. Furthermore, if this procedure fails (that is, k_i cannot be taken as $n - i$ because some limits of quotients when $t \to b$ are infinite) then the space has no STP basis by Theorem 4.7. Summarizing, we have also obtained a test to determine if a space has an STP basis and a method of constructing one if it exists.

5. TCHEBYCHEFF SPACES WITH STRICTLY TOTALLY POSITIVE BASES AND EXTENSIBILITY

In this section we give some important examples of STP spaces. The next result will show that many usual Tchebycheff spaces have totally positive

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bases because, in some Tchebycheff spaces, bicanonical bases can be directly obtained using further properties related to differentiability. First, let us recall the concept of an *extended Tchebycheff system* (ET system): a system of functions $(u_0, ..., u_n)$ in $C^n[a, b]$ is called an ET system if any extended collocation matrix

$$M^*\begin{pmatrix} u_0, ..., u_n \\ t_0, ..., t_n \end{pmatrix} := (L_i[u_j])_{i,j=0,...,n}, \qquad t_0 \leq t_1 \leq \cdots \leq t_n \text{ in } [a, b],$$

has positive determinant, where $L_i[u] := u^{(m_i)}(t_i)$, $m_i = \#\{k < i \mid t_k = t_i\}$. An ET space is a vector space with an ET basis. From the definition it follows that the Hermite interpolation problem on an ET space always has a (unique) solution.

THEOREM 5.1. *If $\mathscr{U}$ is an ET space of functions defined on $[a, b]$, then $\mathscr{U}$ is an STP space.*

Proof. Let $n = \dim \mathscr{U} - 1$. For each $i \in \{0, 1, ..., n\}$, let us define $u_i \in \mathscr{U}$ as the unique solution of the Hermite interpolation problem

$$u^{(j)}(a) = 0, \qquad j = 0, ..., i-1, \tag{5.1.a}$$

$$u^{(i)}(a) = 1, \tag{5.1.b}$$

$$u^{(j)}(b) = 0, \qquad j = 0, ..., n-1-i. \tag{5.1.c}$$

It is easy to show that conditions (5.1) imply that u_i has a zero of multiplicity i at a and a zero of multiplicity $n-i$ at b, for each $i = 0, 1, ..., n$ and so $(u_0, ..., u_n)$ is bicanonical. Conditions (5.1) also guarantee that the functions $u_0, ..., u_n$ are linearly independent, and so $(u_0, ..., u_n)$ is a basis of the T space $\mathscr{U}$.

Let us observe that conditions (5.1.a) and (5.1.b) imply that the functions u_i are positive on an interval $(a, a+\delta)$ for some $\delta > 0$. From Theorem 4.4.(i), it follows that they are positive on (a, b). By Theorem 4.4(ii), $(u_0, ..., u_n)$ is STP on (a, b). Let $A = M\binom{u_0, ..., u_n}{t_0, ..., t_n}$, $t_0 < \cdots < t_n \in [a, b]$. If $t_0 > a$ and $t_n < b$, then A is STP. Otherwise A is TP as follows from the fact that, if $t_0 = a$, the first row is $(1, 0, ..., 0)$ and that, if $t_n = b$, the last row is $(0, ..., 0, u_n(b))$. Thus $(u_0, ..., u_n)$ is TP on $[a, b]$ and, by Proposition 2.3, $\mathscr{U}$ is an STP space. ∎

The previous theorem is closely related to Corollary 1.1 of Chapter 6 of [6] which states that in any extended complete Tchebycheff space there exists an (extended) totally positive basis. The proof of this result was based on two facts: an extended complete Tchebycheff space has a *canonical* extended complete Tchebycheff basis (Remark 1.1 of Chapter 6 of [6]) and any canonical extended complete Tchebycheff system is an extended totally positive system (Theorem 1.1 and Theorem 1.2 of Chapter 6 of [6]). Both consequences of Karlin's results can be compared with Theorem 3.6 and Proposition 3.5 of this paper, which have been obtained in a more general framework.

An important source of examples of spaces with STP bases are T spaces whose domain of definition can be extended. In this sense, we point out two recent papers [13, 15]. In those papers it was shown that, under suitable hypotheses, STP spaces are characterized by the property of extending them to Tchebycheff spaces on a larger domain of definition (Theorem 2.2 of [13], Theorem 1 of [15]). The domains of definition S used in these results satisfy the property (B) of Zielke [17], that is, for any two points $s_1 < s_2$ in S there exists $s \in S$ such that $s_1 < s < s_2$. The next theorem is closely related with those results, but it can be applied to any $D \in \mathscr{D}$ without imposing the property (B) of Zielke. We include some proofs which illustrate some techniques used throughout this paper. We also study the question of constructing STP extensions of an STP space.

Definition 5.2. Let $\tilde{\mathscr{U}}$ be a vector space of functions defined on a totally ordered set $\tilde{S}$. If $S \subseteq \tilde{S}$ with the induced order relation and $\mathscr{U}$ is the space formed by the restrictions of $\tilde{\mathscr{U}}$ to S we say that $\tilde{\mathscr{U}}$ is an extension of the space $\mathscr{U}$ to $\tilde{S}$, provided that $\dim \tilde{\mathscr{U}} = \dim \mathscr{U}$.

Theorem 5.3. *Let $\mathscr{U}$ be a T space of functions defined on a set $D \in \mathscr{D}$ such that $a = \inf D > -\infty$. Then the following properties are equivalent*:

(i) *There exists an extension $\tilde{\mathscr{U}}$ to a set $\{\tau_0, ..., \tau_n\} \cup D$ $(\tau_0 < \cdots < \tau_n < a)$ which is a T space.*

(ii) *$\mathscr{U}$ is an STP space.*

(iii) *There exists an extension $\tilde{\mathscr{U}}$ to the set $(2a - D) \cup \{a\} \cup D$ $(2a - D := \{2a - t \mid t \in D\})$ which is a CT space.*

(iv) *There exists an extension $\tilde{\mathscr{U}}$ to a set $D' \cup D$ (D' an infinite set such that* $\sup D' \leqslant \inf D$), *which is an STP space.*

Proof. (i) $\Rightarrow$ (ii). Since $\tilde{\mathscr{U}}$ is a T space, there exist basic functions for the Lagrange interpolation problem

$$l_i(\tau_j) = \delta_{ij}, \qquad \forall i, j \in \{0, 1, ..., n\}.$$

Let us define $w_i := (-1)^i l_{n-i}$, $i = 0, ..., n$. Let us see that $(w_0, ..., w_n)$ is STP on D. We deduce that $(w_0, ..., w_n)$ is a T system because it is a basis of $\tilde{\mathscr{U}}$ and

$$\det M\begin{pmatrix} w_0, ..., w_n \\ \tau_0, ..., \tau_n \end{pmatrix} = \det((-1)^j \delta_{i, n-j})_{i, j = 0, ..., n} = 1 > 0,$$

where $\delta_{k,l}$ equals 1 if $k=l$ and equals 0 otherwise. It remains to see that any minor $\det M\binom{w_{i_0}, \ldots, w_{i_k}}{t_0, \ldots, t_k}$ with $k<n$ is positive, which follows from the straightforward fact

$$0<\det M\begin{pmatrix} w_0, \ldots, w_n \\ \tau_{j_1}, \ldots, \tau_{j_{n-k}}, t_0, \ldots, t_k \end{pmatrix}=\det M\begin{pmatrix} w_{i_0}, \ldots, w_{i_k} \\ t_0, \ldots, t_k \end{pmatrix},$$

where $j_1, \ldots, j_{n-k}$ are the indices such that,

$$\{i_0, \ldots, i_k\} \cup \{n-j_1, \ldots, n-j_{n-k}\}=\{0, 1, \ldots, n\}.$$

Therefore the restrictions of w_i, $i=0, \ldots, n$, to D form an STP basis of $\mathscr{U}$.

(ii) $\Rightarrow$ (iii). By Corollary 4.8(iii), there exists a bicanonical STP basis $(v_0, \ldots, v_n)$ of $\mathscr{U}$. Let us define

$$\tilde{v}_i(t):=\begin{cases} (-1)^i v_i(2a-t) & \text{if } t \in 2a-D, \\ \delta_{0,i} & \text{if } t=a, \\ v_i(t) & \text{if } t \in D. \end{cases} \tag{5.2}$$

Now we may show that $(\tilde{v}_0, \ldots, \tilde{v}_n)$ is a CT system, that is $(\tilde{v}_0, \ldots, \tilde{v}_k)$ is a T system for each $k=0, \ldots, n$. Taking into account that the restriction of $(\tilde{v}_0, \ldots, \tilde{v}_k)$ to D is a T system, it is sufficient to see that the space of functions generated by $(\tilde{v}_0, \ldots, \tilde{v}_k)$ is a T space. By Lemma 2.4 this condition is equivalent to saying that $S^+(\tilde{v}) \leqslant k$ for each $\tilde{v}:=\sum_{i=0}^k \lambda_i \tilde{v}_i$. If $\lambda_0 \neq 0$ it is clear that $\tilde{v}$ has the same sign to the right and to the left of a because this property holds for $\tilde{v}_0$ and $\lim_{t \to a} \tilde{v}_i(t)/\tilde{v}_0(t)=0$, $i=1, \ldots, n$ (see formula (3.1)). Therefore

$$\begin{aligned} S^+(\tilde{v}) &= S^+(\tilde{v}|_{2a-D})+S^+(\tilde{v}|_D) \\ &= S^+\left(\sum_{i=0}^k (-1)^i \lambda_i v_i\right)+S^+\left(\sum_{i=0}^k \lambda_i v_i\right). \end{aligned}$$

Since $(v_0, \ldots, v_k)$ is STP on D,

$$S^+\left(\sum_{i=0}^k \lambda_i v_i\right) \leqslant S^-(\lambda_0, \lambda_1, \ldots, \lambda_k),$$
$$S^+\left(\sum_{i=0}^k (-1)^i \lambda_i v_i\right) \leqslant S^-(\lambda_0, -\lambda_1, \ldots, (-1)^k \lambda_k).$$

Therefore

$$S^+(\tilde{v}) \leqslant S^-(\lambda_0, \lambda_1, \ldots, \lambda_k)+S^-(\lambda_0, -\lambda_1, \ldots, (-1)^k \lambda_k) \leqslant k.$$

If $\lambda_0=0$ and $\lambda_1 \neq 0$, $\tilde{v}=\sum_{i=1}^k \lambda_i \tilde{v}_i$ then $\tilde{v}(a)=0$ and the strict signs of $\tilde{v}$ in a left neighbourhood and a right neighbourhood of a are different because this property holds for $\tilde{v}_1$ and $\lim_{t \to a} \tilde{v}_i(t)/\tilde{v}_1(t) = 0$, $i = 2, ..., n$ (again by formula (3.1)). Then, reasoning as above we obtain

$$\begin{aligned} S^+(\tilde{v}) &\leqslant S^+(\tilde{v}|_{2a-D}) + S^+(\tilde{v}|_D) + 1 \\ &\leqslant S^-(\lambda_1, ..., \lambda_k) + S^-(-\lambda_1, ..., (-1)^k \lambda_k) + 1 \\ &\leqslant k - 1 + 1 = k \end{aligned}$$

because $(v_1, ..., v_k)$ is STP on D. Finally, if $\lambda_0 = 0$ and $\lambda_1 = 0$, $\tilde{v} = \sum_{i=2}^k \lambda_i \tilde{v}_i$ and

$$S^+(\tilde{v}) \leqslant S^+(\tilde{v}|_{2a-D}) + S^+(\tilde{v}|_D) + 2.$$

Since $(v_2, ..., v_n)$ is STP on D,

$$\begin{aligned} S^+(\tilde{v}) &\leqslant S^-(\lambda_2, ..., \lambda_k) + S^-(\lambda_2, ..., (-1)^k \lambda_k) + 2 \\ &\leqslant k - 2 + 2 = k \end{aligned}$$

and (iii) follows.

(iii) $\Rightarrow$ (iv). Let $\mathscr{V}$ be any complete Tchebycheff extension of $\mathscr{U}$ on $2a - D \cup \{a\} \cup D$ and let $\tau_0 < \cdots < \tau_n \in 2a - D$. Since (i) implies (ii), we know that the space generated by the restriction of the functions of $\mathscr{V}$ to $D' \cup D$, where $D' := [(\tau_n, \infty) \cap (2a - D)] \cup \{a\}$, is an STP space. Let us observe that the set D' is infinite because $\sup(2a - D) \notin 2a - D$.

(iv) $\Rightarrow$ (i). Obvious. ∎

Let us observe that if we apply the construction of the extension in (ii) $\Rightarrow$ (i) in the previous theorem to the monomial basis $(1, t, ..., t^n)$ on $(0, \infty)$, which is STP, we obtain the monomials defined on the whole real line, which is a CT system. This example motivated the mentioned construction.

Now, we show an example of a Tchebycheff space defined on an open and bounded interval which does not have any STP basis. This means that this space has no extensions in the sense stated in Theorem 5.3.

EXAMPLE 5.4. Let $\mathscr{U} = \text{span}\,\{1, \varphi(t)\}$, where $\varphi: t \in (-1, 1) \mapsto t/(1 - t^2) \in \mathbb{R}$. Clearly $(1, \varphi)$ is a CT system and then U is a T space (even a CT space). This space has no STP basis because any nonconstant function of the space has a sign change.

From Theorem 5.1 and Theorem 5.3 we deduce the following result.

COROLLARY 5.5. *Each extended Tchebycheff space $\mathscr{U}$ on $[a, b]$ has an extension $\tilde{\mathscr{U}}$ to any interval $[\alpha, b]$ $(2a - b < \alpha < a)$ which is an STP space.*

Proof. Let $(u_0, ..., u_n)$ be the basis defined in Theorem 5.1 by formulae (5.1). In the proof of Theorem 5.1, we showed that this basis is bicanonical and STP on (a, b). Now by (ii) ⇒ (iii) of Theorem (5.3) we obtain that the extension

$$\tilde{u}_i(t) := \begin{cases} (-1)^i u_i(2a-t) & \text{if } t \in [2a-b, a), \\ u_i(t) & \text{if } t \in [a, b], \end{cases}$$

is CT on $(2a-b, b)$. In particular $(\tilde{u}_0, ..., \tilde{u}_n)$ is a T system on $(2a-b, b)$ and taking into account that the last column of

$$M\begin{pmatrix} \tilde{u}_0, ..., \tilde{u}_n \\ t_0, ..., t_{n-1}, b \end{pmatrix}, \qquad 2a-b < t_0 < \cdots < t_{n-1} < b,$$

is $(0, ..., 0, 1)^T$ and that $(\tilde{u}_0, ..., \tilde{u}_{n-1})$ is a T system on $(2a-b, b)$, we deduce that $(\tilde{u}_0, ..., \tilde{u}_n)$ is a T system on $(2a-b, b]$.

On the other hand, taking $2a-b < \tau_0 < \cdots < \tau_n < \alpha$ and following the same steps of the proof (i) ⇒ (ii) of Theorem 5.3 we conclude that the space generated by $\tilde{u}_0, ..., \tilde{u}_n$ is STP on $[\alpha, b]$. ∎

ACKNOWLEDGMENTS

We thank the referees for pointing out to us the papers [13, 15], which were not referenced in an earlier version of the manuscript.

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